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Conservation laws and universality in branching annihilating random walks

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Abstract. Recently, Takayasu and Tretyakov studied branching annihilating random walks with $n = 1-5$ offspring. These models exhibit a continuous phase transition to an absorbing state. For odd n the models belong to the universality class of directed percolation. For even n the particle number is conserved modulo two and the critical behaviour is not compatible with directed percolation. In this paper branching annihilating random walks with $n = 4$ plus an additional process (spontaneous annihilation) which breaks the conservation law are studied. The inclusion of spontaneous annihilation, even at very small rates, leads to directed percolation critical behaviour.

1. Introduction

Recently, Takayasu and Tretyakov [1], studied the branching annihilating random walk (BAW) with $n = 1-5$ offspring. In the BAW a particle is chosen at random. With probability p it jumps to a randomly chosen nearest neighbour and if this site is already occupied both particles are annihilated. With probability $1 - p$ the particle produces n offspring, which are placed on the closest neighbouring sites. When an offspring is created on a site which is already occupied it annihilates with the occupying particle leaving an empty site. In one dimension for $n = 1$ it has been shown [2] that the BAW has an active steady-state for sufficiently small p . Computer simulations revealed that the phase transition from the active state to the absorbing state is continuous [1]. When n is odd there is a *unique* absorbing state. This type of phase transition has been studied in numerous other models such as the contact process [3–5], Schlögl's first and second models [6–9], directed percolation (DP) [10–12] and Reggeon field theory (RFT) [7, 13]. Studies of related models demonstrate the robustness of this universality class against a wide range of changes in the local kinetic rules, such as multi-particle processes [14–16], diffusion [17] and changes in the number of components [18]. BAWs with an odd number of offspring include a combination of diffusion and various multi-particle processes. One would therefore expect, bearing in mind the robustness of DP critical behaviour, that the transition should belong to the universality class of directed percolation. The steady-state concentration of particles $\bar{\rho}$ (which is the appropriate order parameter) decays as

$$\bar{\rho} \propto |p_c - p|^\beta \quad (1)$$

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where β is the order parameter critical exponent. Estimates for β , obtained from computer simulations, were however only marginally consistent with directed percolation. Takayasu and Tretyakov [1] found $p_c = 0.108 \pm 0.001$ and $\beta = 0.32 \pm 0.01$ which should be compared to the value $\beta^{\text{DP}} = 0.2769 \pm 0.0002$ [19] for directed percolation in 1 + 1 dimensions. For $n = 3$ and $n = 5$ they found that $p_c = 0.461 \pm 0.002$ and 0.718 ± 0.001 , respectively, with $\beta = 0.33 \pm 0.01$ in both cases. Time-dependent computer simulations [20] for $n = 1$ and 3 yielded estimates for three critical exponents in good agreement with directed percolation, thus supporting the notion that BAWs with an odd number of offspring belong to the DP universality class.

For $n = 2$ the model does not have an active steady state [21] whereas for $n = 4$ it was found [1] that $\beta = 0.7(1)$, and the model does not belong to the DP universality class. Grassberger *et al* [22] studied a model, involving the processes $X \rightarrow 3X$ and $2X \rightarrow 0$, very similar to the BAW with $n = 2$. Steady-state and time-dependent computer simulations, yielded non-DP values for various critical exponents. Note that in both the model proposed by Grassberger *et al* and in BAWs with an even number of offspring the number of particles is conserved modulo two. This conservation law might be responsible for the non-DP behaviour. With this in mind a modified version of the BAW with $n = 4$ is studied. If the conservation of particle number modulo two is indeed responsible for the non-DP behaviour, breaking this conservation law should produce DP exponents. The easiest way to obtain this is probably by adding spontaneous annihilation of particles. To be more precise we study a model in which particles diffuse, with probability p_d , according to the BAW rules, are annihilated spontaneously with probability $(1 - p_d)p_a$, or else create four offspring following the BAW rules. For $p_a = 0$ we thus recover the BAW with $n = 4$.

2. Time-dependent behaviour

In this paper we present results from computer simulations of the one-dimensional BAW with $n = 4$ and spontaneous annihilation using time-dependent simulation and finite-size scaling. Earlier studies [7, 12, 15, 16, 23] have revealed that time-dependent simulation is a very effective method for locating critical points and estimating exponents. In time-dependent simulations we start from a configuration close to the absorbing state, and then follow the 'average' time evolution of this configuration by simulating a large ensemble of independent realizations. In the simulations presented here we always started, at $t = 0$, with an empty lattice except for two occupied nearest neighbours at the origin. We then performed a number, N_S , of independent runs, typically 50 000, for different values of p_d in the vicinity of the critical point p_d^c . The value of p_a remained fixed in each set of simulations. As the number of particles is very small an efficient algorithm may be devised by keeping a list of occupied sites. In each elementary step a particle is drawn at random from this list and the processes are performed according to the rules given earlier. Before each elementary change the time variable is incremented by $1/n(t)$, where $n(t)$ is the number of particles prior to the change. Thus one time step equals (on average) one attempted update per lattice site. Each run had a maximal duration of t_M time steps. I measured the survival probability $P(t)$ (the probability that the system had not entered the absorbing state at time t), the average number of occupied sites $\bar{n}(t)$ and the average mean square distance of spreading $\bar{R}^2(t)$ from the centre of the lattice. Notice that $\bar{n}(t)$ is averaged over all runs whereas $\bar{R}^2(t)$ is averaged only over the surviving runs. From the scaling ansatz for the contact process and similar models [7, 12] it follows that the quantities defined above are governed by power

laws at p_d^c as $t \rightarrow \infty$

$$P(t) \propto t^{-\delta} \tag{2}$$

$$\bar{n}(t) \propto t^\eta \tag{3}$$

$$\bar{R}^2(t) \propto t^z. \tag{4}$$

In log-log plots of $P(t)$, $\bar{n}(t)$ and $\bar{R}^2(t)$ against t we should see asymptotically a straight line at $p_d = p_d^c$. The curves will show positive (negative) curvature when $p_d < p_d^c$ ($p_d > p_d^c$). This makes it possible to obtain accurate estimates for p_d^c . The asymptotic slope of the (critical) curves define the dynamic critical exponents δ , η and z . Generally one has to expect corrections to the pure power law behaviour so that $P(t)$ is more accurately given by [12]

$$P(t) \propto t^{-\delta}(1 + at^{-1} + bt^{-\delta'} + \dots) \tag{5}$$

and similarly for $\bar{n}(t)$ and $\bar{R}^2(t)$. More precise estimates for the critical exponents can be obtained if one looks at local slopes

$$-\delta(t) = \frac{\ln[P(t)/P(t/m)]}{\ln(m)} \tag{6}$$

and similarly for $\eta(t)$ and $z(t)$; in this work we used $m = 5$. The local slope $\delta(t)$ behaves as [12]

$$\delta(t) = \delta + at^{-1} + b\delta' t^{-\delta'} + \dots \tag{7}$$

with similar expressions applying for $\eta(t)$ and $z(t)$. In a plot of the local slopes against $1/t$ the critical exponents are given by the intercept of the curve for p_d^c with the y axis. The off-critical curves often have a very notable curvature, i.e. one will see the curves for $p_d > p_d^c$ veering downward while the curves for $p_d < p_d^c$ veer upward.

2.1. Results from computer simulations

In figure 1 the local slopes $-\delta(t)$, $\eta(t)$ and $z(t)$ for $p_a = 0.1$ are plotted. In these simulations the maximal number of timesteps $t_M = 10\,000$ and the number of independent runs $N_S = 50\,000$. From these results we see that the curve for $\eta(t)$ at $p_a = 0.491\,00$ has a notable upwards curvature when $1/t \rightarrow 0$. Likewise we see that the curve for $p_a = 0.491\,10$ veers downward, leading to the very precise estimate $p_d^c = 0.491\,05 \pm 0.000\,05$. From the intercept of the 'critical' curves with the y axis we estimate $\delta = 0.161 \pm 0.002$, $\eta = 0.312 \pm 0.005$ and $z = 1.26 \pm 0.01$. These values for δ , η and z are in excellent agreement with the values obtained from series expansions for the contact process and related models [19]: $\delta = 0.1597 \pm 0.0003$, $\eta = 0.314 \pm 0.003$ and $z = 1.266 \pm 0.007$. As a further test of the consistency of the data one may use the scaling relation [7]

$$\delta = \frac{1}{2} \left(\frac{d}{2} z - \eta \right). \tag{8}$$

It is clearly seen that the estimates for the BAW given above agree very well with this scaling relation.

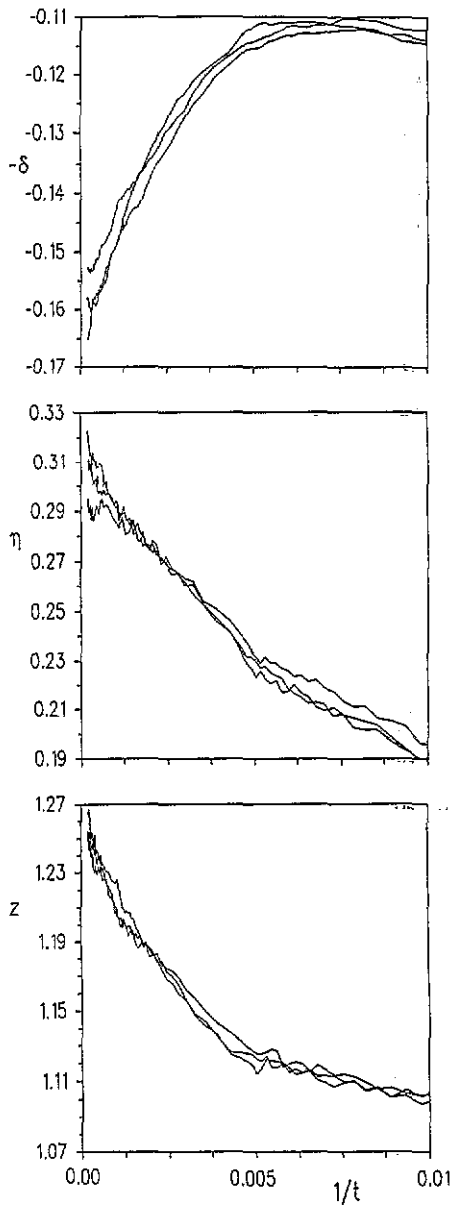


Figure 1. Local slopes $-\delta(t)$ (upper panel), $\eta(t)$ (middle panel) and $z(t)$ (lower panel), as defined in (6) with $m = 5$, for $p_a = 0.1$. Each panel contains three curves with, from bottom to top, $p_a = 0.49110$, 0.49105 and 0.49100 .

Table 1. Estimates for the location of the critical point p_a^c and the critical exponents δ , η and z as obtained from time-dependent simulations for various values of the spontaneous annihilation probability p_a .

p_a	p_a^c	δ	η	z
0.1	0.49105(5)	0.161(2)	0.312(5)	1.26(1)
0.01	0.6455(5)	0.162(4)	0.312(8)	1.265(15)
0.001	0.6830(5)	0.155(10)	0.31(1)	1.25(2)
0.0001	0.700(5)	0.15(1)	0.305(15)	1.23(3)

We have just seen that the BAW with $n = 4$ and spontaneous annihilation belongs to

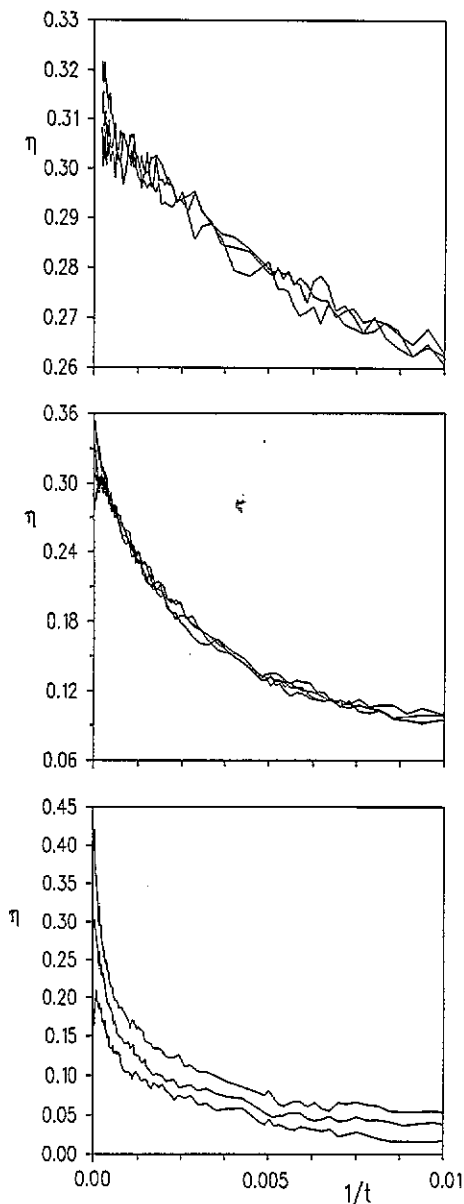
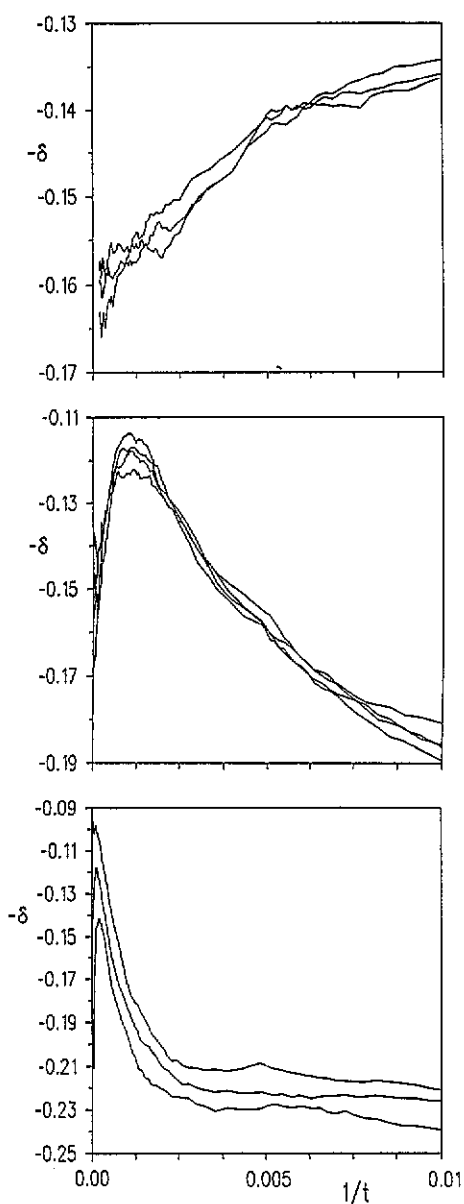


Figure 2. Local slopes $-\delta(t)$, as defined in (6) with $m = 5$, for $p_a = 0.01$ (upper panel), 0.001 (middle panel) and 0.0001 (lower panel). From bottom to top the curves corresponds to $p_d = 0.6460, 0.6455$ and 0.6450 in the upper panel, $p_d = 0.6835, 0.6830, 0.6825$ and 0.6820 in the middle panel, and finally $p_d = 0.705, 0.700$ and 0.695 in the lower panel.

Figure 3. Same as in figure 2 but for the local slopes $\eta(t)$.

the DP universality class when the annihilation probability is relatively high. Three lower rates of spontaneous annihilation were also checked. In table 1 are listed the estimates for the critical point, p_d^c , and the critical exponents δ, η and z as obtained from time-dependent

simulations with $p_a = 0.1, 0.01, 0.001$ and 0.0001 . In these simulations we averaged over $N_S = 50\,000$ independent samples for each value of p_d , and the maximal duration of each run was $t_M = 10\,000$ for $p_a = 0.1$ and 0.01 , $t_M = 25\,000$ for $p_a = 0.001$ and 0.0001 . The values for the critical exponents show that the BAW belongs to the DP universality class even when p_a is as low as 0.0001 . One noticeable difference between the various values for p_a is the increased uncertainty in the estimates. As p_a decreases the short-time transient behaviour becomes more prominent and the estimation of the critical behaviour harder. This is clearly seen in figures 2 and 3 which depict the local slopes $-\delta(t)$ and $\eta(t)$ for $p_a = 0.01, 0.001$ and 0.0001 . As can be seen the local slopes exhibit pronounced short-time deviations from the DP values, which are however reached asymptotically at p_d^c . In particular $-\delta(t)$ starts off below the DP value, which is clearly seen for $p_a = 0.0001$, then over-shoots before the asymptotic DP behaviour finally takes over at p_d^c . $\eta(t)$ starts off below the DP value but eventually increases and reaches the DP value. For $p_a = 0.0001$ these short-time transients are very strong as both $-\delta(t)$ and $\eta(t)$ are initially quite stable at values well below the DP exponents. The off-critical curves exhibit a similar initial behaviour, however the asymptotic behaviour is different. For $p_d < p_d^c$ ($p_d > p_d^c$) we expect to see the local slopes veer upwards (downwards). A closer look at figures 2 and 3 reveals that for $p_a = 0.01$ the curve for $p_d = 0.6460$ veers downward while the curve for $p_d = 0.6450$ veers upward, leading to the estimate $p_d^c = 0.6455 \pm 0.0005$. Similarly we see that $p_d^c = 0.6830 \pm 0.0005$ for $p_a = 0.001$ and $p_d^c = 0.700 \pm 0.005$ for $p_a = 0.0001$. My estimates for the corresponding critical exponents, all of which are consistent with DP critical behaviour, are listed in table I.

The similarity in the behaviour of the simulation results for the various values of p_a is quite reassuring especially since the short-time effects become so pronounced when p_a is decreased. The short-time transient behaviour for small p_a , though very prominent, cannot totally obscure the true asymptotic behaviour. It is thus with a great deal of confidence that we conclude that the time-dependent simulations indicate that the BAW with $n = 4$ and spontaneous annihilation belongs to the DP universality class for all values of $p_a \geq 0.0001$. In the next section we will show that this conclusion is supported by results from a finite-size scaling analysis.

3. Finite-size scaling analysis

The concepts of finite-size scaling [24, 25], though originally developed for equilibrium systems, are also applicable to non-equilibrium second-order phase transitions. Aukrust *et al* [26] showed how finite-size scaling can be used very successfully to study the critical behaviour of non-equilibrium systems exhibiting a continuous phase transition to an absorbing state. Their method was later applied to models with infinitely many absorbing states [27]. As in equilibrium second-order phase transitions one assumes that the (infinite-size) non-equilibrium system features a length scale which diverges at criticality as

$$\xi(p) \propto |p_d^c - p_d|^{-\nu_\perp} \quad (9)$$

where ν_\perp is the correlation length exponent in the space direction. We expect finite-size effects to become important when the correlation length $\xi(p) \sim L$. The basic finite-size scaling ansatz is that the various quantities depend on system-size only through the scaled length $L/\xi(p)$, or equivalently through the variable $|p_d^c - p_d|L^{1/\nu_\perp}$. Thus we assume that the order parameter depends on system size and distance from the critical point as

$$\rho_a(p_d, L) \propto L^{-\beta/\nu_\perp} f((p_d^c - p_d)L^{1/\nu_\perp}) \quad (10)$$

such that at p_d^c

$$\rho_s(p_d^c, L) \propto L^{-\beta/\nu_\perp} \tag{11}$$

and

$$f(x) \propto x^\beta \quad \text{for } x \rightarrow \infty \tag{12}$$

so that (1) is recovered when $L \rightarrow \infty$ in the critical region. In ρ_s , and other quantities, the subscript s indicates an average taken over the *surviving* samples, i.e. the average includes only those samples which have not yet entered the absorbing state. The restriction to surviving samples is quite natural and ensures that a quantity such as ρ_s becomes constant after a relative short transient time [26, 27]. Once this *quasisteady state* has been reached one may perform the usual averages. Figure 4 is a log-log plot of the average concentration of particles $\rho_s(p_d^c, L)$ in the quasisteady state as a function of L at the critical point for the various values of p_a studied earlier. The number of timesteps t_M and independent samples N_S varied from $t = 500, N = 50\,000$ for $L = 16$ to $t_M = 1\,000\,000, N = 100$ for $L = 4096$. The slopes of the critical curves lie in the interval $\beta/\nu_\perp = 0.252\text{--}0.259$, which is in very good agreement with the standard DP value 0.2523 ± 0.0004 as obtained from the estimates $\beta = 0.2769(2)$ [19] and $\nu_\perp = 1.0972(6)$ [28]. So these results confirm that the model belongs to the DP universality class. Note however that while the asymptotic (large L) behaviour is DP-like, for small L the results show a very distinct deviation from the power law. These deviations become more pronounced as p_a is decreased.

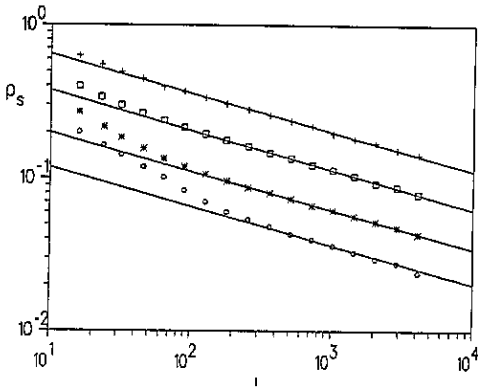


Figure 4. Log-log plot of $\rho_s(p_d^c, L)$ against L for, from top to bottom, $p_a = 0.1, 0.01, 0.001$ and 0.0001 , with the corresponding values for the critical point $p_d^c = 0.491\,05, 0.6455, 0.6830, 0.700$. The results for some of the p_a values have been scaled so as to ensure a clear separation of the data points and thus make it easy to see the behaviour of each set of data.

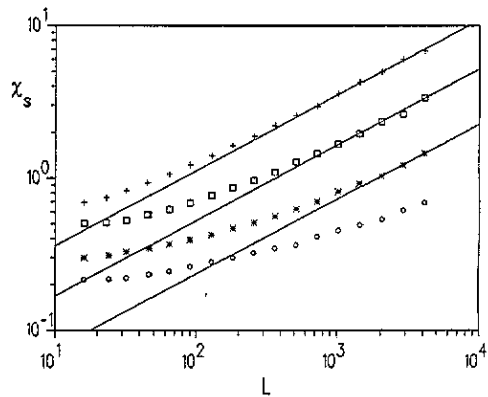


Figure 5. Log-log plot of $\chi_s(p_d^c, L)$ against L (see the caption to figure 4 for details).

Another exponent estimate can be obtained using the generalized susceptibility, which in the steady-state is defined as

$$\bar{\chi} = L^d (\langle \rho^2 \rangle - \langle \rho \rangle^2) \propto |p_d^c - p_d|^{-\nu} \tag{13}$$

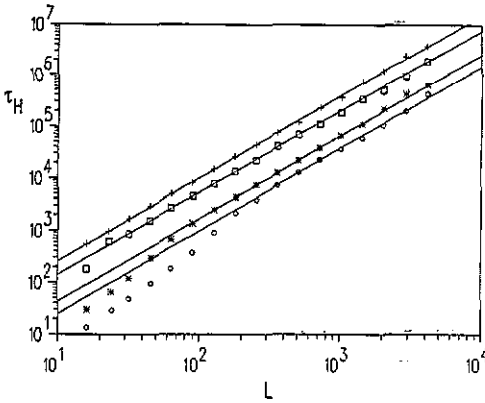


Figure 6. Log-log plot of $\tau_s(p_d^c, L)$ against L (see the caption to figure 4 for details).

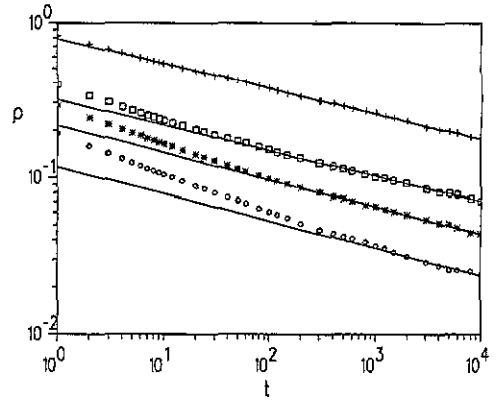


Figure 7. Log-log plot of the short-time decay of $\rho_s(p_d^c)$, with $L = 8192$, for, from top to bottom, $p_a = 0.1, 0.01, 0.001$ and 0.0001 .

where L is the linear extension of the system. $\bar{\chi}$ is a quantity analogous to the susceptibility as defined for equilibrium magnetic systems. Actually $\bar{\chi}$ is just a measure of the typical size of fluctuations. Equation (13) thus leads to the following finite-size scaling ansatz

$$\chi_s(p_d, L) \propto L^{\gamma/\nu_\perp} g((p_d^c - p_d)L^{1/\nu_\perp}) \quad (14)$$

and

$$\chi_s(p_d^c, L) \propto L^{\gamma/\nu_\perp}. \quad (15)$$

Figure 5 shows a log-log plot of the susceptibility $\chi_s(p_d^c, L)$ as a function of L . The slopes of the lines are $\gamma/\nu_\perp \simeq 0.50$ which is in excellent agreement with the DP value $0.496(2)$ as obtained from the estimate $\gamma = 0.544 \pm 0.001$ [28]†. Again we see that the large L behaviour is DP-like. But this time the deviations for small L are more severe, and in particular we see that for $p_a = 0.0001$ DP behaviour has not been attained even for $L = 4096$.

Additional exponents may be obtained from the dynamical behaviour of the system. In this study we define a characteristic time, $\tau(p_d, L)$, as the time it takes for half the samples to enter the absorbing state. In general one has to expect that the characteristic time diverges as $p_d \rightarrow p_d^c$ from above in the steady-state

$$\tau(p) \propto |p_d^c - p|^{-\nu_\parallel} \quad (16)$$

where ν_\parallel is the correlation length exponent in the time direction. This leads to the following finite-size scaling form

$$\tau(p_d, L) \propto L^y h((p_d^c - p_d)L^{1/\nu_\perp}) \quad (17)$$

where $y = \nu_\parallel/\nu_\perp$. At p_d^c we thus have

$$\tau(p_d^c, L) \propto L^y. \quad (18)$$

In figure 6 is plotted, on a log-log scale, $\tau_s(p_d^c, L)$ as a function of L . The slopes of the lines yield estimates of ν_\parallel/ν_\perp in the range 1.57 – 1.59 which again agrees perfectly with the DP value $1.579(2)$ as obtained from the estimate $\nu_\parallel = 1.733 \pm 0.001$ [28]. Again we see that the large L behaviour is DP-like. The deviations for small L are less severe.

† The notation for the critical exponents differs from that of directed percolation. β , ν_\parallel and ν_\perp are the same, but $\gamma^{\text{DP}} = \gamma + \nu_\parallel + (1-d)\nu_\perp$. We used the latter relation to obtain the estimate for γ .

One may also study the dynamical behaviour by looking at the time dependence of $\rho_s(p_d^c, L, t)$. For $t \gg 1$ and $L \gg 1$ one can assume a scaling form

$$\rho_s(p_d^c, L, t) \propto L^{-\beta/\nu_\perp} f(t/L^\gamma). \quad (19)$$

At p_d^c the system shows a power law behaviour for $t < L^\gamma$ before finite-size effects become important. Thus for $L \gg 1$ and $t < L^\gamma$, $\rho(p_d^c, L, t) \propto t^{-\theta}$. From (19) we see that this is the case for large L only if the scaling relation

$$\theta = \beta/(\nu_\perp \gamma) = \beta/\nu_\parallel \quad (20)$$

holds. Figure 7 shows the short-time evolution of the concentration of particles at p_d^c for the various values of p_a studied in this article. The asymptotic behaviour yields the estimate $\theta \simeq 0.16$ for $p_a = 0.1$ and 0.01 , and $\theta \simeq 0.17$ for $p_a = 0.001$ and 0.0001 . These estimates agree pretty well with the value for directed percolation $\theta = \beta/\nu_\parallel = 0.1598(3)$, as obtained from the estimates cited above for β and ν_\parallel .

4. Summary and discussion

In this article the results from a study of branching annihilating walks with four offspring and spontaneous annihilation of particles are reported. The regular BAW with four offspring exhibits a critical behaviour that is distinctly different from that of directed percolation. The addition of spontaneous annihilation changes this critical behaviour even when p_a is as low as 0.0001 . I have obtained several independent exponent estimates, using time-dependent simulations and finite-size scaling, all of which are consistent with DP critical behaviour. This is very firm evidence that the BAW with four offspring and spontaneous annihilation belongs to the DP universality class. The non-DP behaviour of the BAW with four offspring and the model proposed by Grassberger *et al* [22] is probably due to the conservation of particle number modulo two. The fact that breaking this conservation law, by adding spontaneous annihilation, results in DP critical behaviour certainly supports this notion.

While we find DP behaviour asymptotically for all values of p_a studied there are some very pronounced short-time and small-size effects. These effects become more important when p_a is decreased. In particular for $p_a = 0.0001$ these effects are so strong that some of the methods used to obtain exponent estimates either fail or yield very poor estimates.

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